A BAYES SOLUTION FOR THE PROBLEM OF RANKING POISSON PARAMETERS

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Summary

It is desired to rank the parameters from a set of Poisson populations based on fixed, equal size samples from each population. A Bayes solution is derived for several types of loss functions and gamma priors, under the usual assumptions of symmetric and additive losses and symmetric priors.

1. Introduction

We assume that a set of n Poisson populations has been observed, yielding (through reduction by sufficiency) the data $x_1, \cdots, x_n$. (The results presented apply equally as well to a set of Poisson processes observed over time periods of equal length.) The object is to determine an ordering of the parameters $\lambda_1, \cdots, \lambda_n$, allowing the possibility of "ties". That is, for each pair of parameters λ_i, λ_j, there are three decisions available: λ_i is greater than λ_j, λ_j is greater than λ_i, λ_i and λ_j are unranked. The approach we adopt is thus closely related to that of earlier work by Duncan [2], [3], Bland and Bratcher [1], and Naik [5].

2. General description and solution of the problem

We make the following basic assumptions:

(i) The prior information, in the form of a probability distribution, is the same for each λ_i. We denote the prior by $\varphi(\lambda)$.

(ii) The loss function for the overall problem can be taken to be the sum of the loss functions for certain component problems.

(iii) Finally, the loss functions for the component problems are symmetric in the sense that permutations of the decisions and corresponding parameters leave the loss unchanged.

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The assumptions (i) and (iii) can best be summarized and justified by saying that the populations are to be treated equally *a priori* and the seriousness of an incorrect decision is the same regardless of the populations concerned. The concept of additive loss, which appears in assumption (ii), was introduced and discussed by Lehmann [4].

The component problems mentioned above are two-decision problems which are easily solved. In particular, for each pair of parameters, say λ and μ, there are two component two-decision problems related to the three-decision problem described in Section 1. One consists of the decisions d_0: λ and μ are unranked and d_1: $\lambda > \mu$. The other problem is obtained by interchanging the parameters. Denote by $L(d; \lambda, \mu)$ the loss incurred in making decision d when λ and μ are the true values of the parameters.

It is seen that a solution to each of the component two-decision problems for λ and μ will specify a solution to the three-decision problem, provided only that the solutions to the component problems are compatible. That is, we require that it is never possible to make the two decisions $\lambda > \mu$ and $\mu > \lambda$ simultaneously. In the same way, solutions to the three-decision problems lead to a solution for the overall problem, if inconsistent decisions (such as $\lambda_1 > \lambda_2$, $\lambda_2 > \lambda_3$, and $\lambda_3 > \lambda_1$) are never made. We do, however, allow decisions of the type: λ_1 and λ_2 are unranked, λ_2 and λ_3 are unranked, but $\lambda_1 > \lambda_3$.

Because of the assumptions of symmetric losses and identical priors only one two-decision problem actually need be solved; these assumptions are not essential and the problem could be solved without them.

THEOREM. *Under assumptions* (i), (ii), *and* (iii), *the Bayes rule for the overall multiple comparisons problem is given by the following*: *for each pair* λ_i, λ_j, *make decision* d_1 *if* $h(x_i, x_j) > 0$; *otherwise, make decision* d_0, *where*

$$h(x_i, x_j) = \int_0^\infty \int_0^\infty [L(d_0; \lambda_i, \lambda_j) - L(d_1; \lambda_i, \lambda_j)] \cdot f(x_i \mid \lambda_i) f(x_j \mid \lambda_j) \varphi(\lambda_i) \varphi(\lambda_j) d\lambda_i d\lambda_j ,$$

provided these component solutions are compatible. ($f(x \mid \lambda)$ *is the probability mass function of the observed Poisson variables.*) *Note that the solution is unique if and only if the probability that* $h(x_i, x_j) = 0$ *is zero.*

We omit the proof; it is tedious algebraically and is available for similar situations, along with further discussion and background, in the references cited above.

3. The Bayes rule for gamma priors and absolute error loss

Suppose that we observe X and Y from Poisson populations with parameters λ and μ, respectively. We consider first the loss function given by

$$
(1)\qquad
\begin{aligned}
L(d_0;\lambda,\mu) &= \begin{cases} 0\,, & \text{if } \lambda \leqq \mu \\ k_0(\lambda-\mu)\,, & \text{if } \lambda > \mu \end{cases} \\
L(d_1;\lambda,\mu) &= \begin{cases} k_1(\mu-\lambda)\,, & \text{if } \lambda \leqq \mu \\ 0\,, & \text{if } \lambda > \mu \end{cases}
\end{aligned}
$$

where $k_1 \geqq k_0 > 0$ are known, and the prior density function is

$$
\varphi(\lambda) = \begin{cases} \beta^{\alpha+1}\lambda^{\alpha}e^{-\beta\lambda}/\Gamma(\alpha+1)\,, & \text{if } \lambda>0 \ (\alpha>-1,\ \beta>0) \\ 0\,, & \text{if } \lambda \leqq 0 \end{cases}
$$

where α and β are assumed known.

Then, writing $(u)^+$ for the positive part of u,

$$
\begin{aligned}
h(x, y) = \int_0^\infty \int_0^\infty & [k_0(\lambda-\mu)^+ - k_1(\mu-\lambda)^+][x!y!\Gamma(\alpha+1)\Gamma(\alpha+1)]^{-1}\lambda^{x+\alpha}\mu^{y+\alpha} \\
& \cdot \exp[-(1+\beta)(\lambda+\mu)]d\lambda d\mu\,.
\end{aligned}
$$

Since we are interested only in the sign of $h(x, y)$, we can freely neglect *positive* multiples, even those involving x and y. We will use the symbol "$\propto$" in this sense only. We introduce the variable $r=\mu/\lambda$ and let $k=k_1/k_0$. Then

$$
\begin{aligned}
h(x, y) &\propto \int_0^\infty \int_0^\infty [\lambda(1-r)^+ - k\lambda(r-1)^+]\lambda^{x+y+2\alpha}r^{y+\alpha} \exp[-\lambda(1+\beta)(1+r)]\lambda d\lambda dr \\
&= \int_0^\infty [(1-r)^+ - k(r-1)^+]r^{y+\alpha} \int_0^\infty \lambda^{x+y+2\alpha+2} \exp[-\lambda(1+\beta)(1+r)]d\lambda dr \\
&\propto \int_0^\infty [(1-r)^+ - k(r-1)^+]r^{y+\alpha}(1+r)^{-(x+y+2\alpha+3)}dr \\
&= \int_0^1 (1-r)r^{y+\alpha}(1+r)^{-(x+y+2\alpha+3)}dr \\
&\quad - k\int_1^\infty (r-1)r^{y+\alpha}(1+r)^{-(x+y+2\alpha+3)}dr\,.
\end{aligned}
$$

Changing from r to $1/r$ in the second integral and then to $z=r/(1+r)$ in both, we get

$$
(2)\qquad
\begin{aligned}
h(x, y) \propto & \int_0^{1/2} (1-2z)z^{y+\alpha}(1-z)^{x+\alpha}dz \\
& - k\int_0^{1/2} (1-2z)z^{x+\alpha}(1-z)^{y+\alpha}dz\,.
\end{aligned}
$$

Now let $I_{1/2}(m, n)=\Gamma(m+n+2)[\Gamma(m+1)\Gamma(n+1)]^{-1}\int_0^{1/2} z^m(1-z)^n dz$, the incomplete beta integral. Making use of the identity $I_{1/2}(m, n)=1-I_{1/2}(n, m)$, we find that

$$\begin{aligned} h(x, y)\propto x-y-(k+1)(x+y+2\alpha+2)I_{1/2}(x+\alpha, y+\alpha) \\ +2(y+\alpha+1)I_{1/2}(x+\alpha, y+\alpha+1) \\ +2k(x+\alpha+1)I_{1/2}(x+\alpha+1, y+\alpha) . \end{aligned}$$

Finally,

$$\begin{aligned} I_{1/2}(m, n)=I_{1/2}(m-1, n+1)-\Gamma(m+n+2) \\ \cdot [\Gamma(m-1)\Gamma(n+2)]^{-1/2-(m+n+1)} , \end{aligned}$$

so

$$\begin{aligned} h(x, y)\propto x-y-k\Gamma(x+y+2\alpha+3)[\Gamma(x+\alpha+1)\Gamma(y+\alpha+2)2^{x+y+2\alpha+1}]^{-1} \\ +2[kx+y+(k+1)(\alpha+1)]I_{1/2}(x+\alpha, y+\alpha+1) \\ -(k+1)(x+y+2\alpha+2)I_{1/2}(x+\alpha, y+\alpha) . \end{aligned}$$

It remains to show that the component solutions are compatible. We can write (2) as

$$\begin{aligned} \int_0^{1/2} (1-2z)z^{x+\alpha}(1-z)^{x+\alpha}[z^{y-x}-k(1-z)^{y-x}]dz \\ =\int_0^{1/2} (1-2z)z^{y+\alpha}(1-z)^{x+\alpha}\left[1-k\left(\frac{1-z}{z}\right)^{y-x}\right]dz . \end{aligned}$$

The last factor in the integrand is clearly non-positive for $x\leqq y$ (since k is assumed not less than one) whereas the other factors are positive. So for $x\leqq y$, $h(x, y)\leqq 0$ and decision d_0 is made. That is, the set $\{(x, y): h(x, y)>0\}$ is contained in the set $\{(x, y): x>y\}$. In the other component problem involving λ and μ, the decision $\mu>\lambda$ would be made when $h(y, x)>0$. But by symmetry, the set $\{(x, y): h(x, y)>0\}$ is a subset of $\{(x, y): y>x\}$. Hence the sets $\{(x, y): h(x, y)>0\}$ and $\{(x, y): h(y, x)>0\}$ do not intersect, so the solutions given by the component two-decision problems are compatible. The compatibility condition is seen to be that the ratio k_1/k_0 is not less than unity.

4. Other loss functions

If $|\lambda-\mu|$ is replaced in (1) by $|\lambda-\mu|^p$, for any non-negative integer p, the same sequence of steps gives

(3) $$h(x, y)\propto\sum_{j=0}^{p}\binom{p}{j}(-2)^j[\Gamma(x+y+2\alpha+j+2)]^{-1}$$

$$\cdot\,[\Gamma(x+y+1)\Gamma(y+\alpha+j+1)I_{1/2}(y+\alpha+j,\, x+\alpha)$$
$$-k\Gamma(x+\alpha+j+1)\Gamma(y+\alpha+1)I_{1/2}(x+\alpha+j,\, y+\alpha)]\ .$$

Another form of loss function is obtained by substituting $(\lambda/\mu)^{\gamma}$ for $(\lambda-\mu)$ and $(\mu/\lambda)^{\gamma}$ for $(\mu-\lambda)$ for any $0\leqq\gamma\leqq\alpha+1$. Then it is easily verified that

$$(4)\qquad h(x,y)\propto\Gamma(x+\alpha+\gamma+1)\Gamma(y+\alpha-\gamma+1)I_{1/2}(y+\alpha-\gamma,\, x+\alpha+\gamma)$$
$$-k\Gamma(x+\alpha-\gamma+1)\Gamma(y+\alpha+\gamma+1)I_{1/2}(x+\alpha-\gamma,\, y+\alpha+\gamma)\ .$$

In the special case of (3) with $p=0$ or of (4) with $\gamma=0$ we get

$$h(x,y)\propto 1-(k+1)I_{1/2}(x+\alpha,\, y+\alpha)\ .$$

The same compatibility condition applies in these cases.

5. Form of the critical region

It is of interest to verify that for fixed y, we make decision d_0 for $x\leqq x_0(y)$ and decision d_1 otherwise. This result is not obvious since $h(x,y)$ is not monotonic in x for fixed y; the result is true, however, for the cases considered in the previous section, as can be established by the following reasoning. The form of equation (2) corresponding to the loss function involving $(\lambda/\mu)^{\gamma}$ and $(\mu/\lambda)^{\gamma}$ is

$$\int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}[(1-z)^{x-y}-kz^{x-y}]dz\ .$$

Suppose that for some given x and y this expression, a positive multiple of $h(x,y)$, is negative. Then $h(x,y)<0$ and $h(x-1,y)$ is a positive multiple of

$$\int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}[(1-z)^{x-y-1}-kz^{x-y-1}]dz$$
$$\leqq\int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}[2(1-z)^{x-y}-2kz^{x-y}]dz$$
$$=2\int_0^{1/2} z^{y+\alpha-\gamma}(1-z)^{y+\alpha+\gamma}[(1-z)^{x-y}-kz^{x-y}]dz\ .$$

Therefore, $h(x-1,y)$ is negative whenever $h(x,y)$ is negative. An argument along the same lines shows that for the loss function involving a power of the absolute difference of the parameters the same statement holds. Furthermore, for either type of loss, $h(x,y)$ positive implies that $h(x+1,y)$ is positive. It follows that for fixed y, $h(x,y)$ changes sign at most once.

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